High-gain Observer-based Estimation of Parameter Variations with Delay Alignment

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Abstract—This paper analyzes the estimation delay in a high gain observer, where the state estimates may lag behind the actual states due to the observer's non-zero phase response. The paper proves that, for a slowly time-varying system subject to bounded noises, the estimation delay depends on the observer gain, but is independent of the variations of system parameters. Rather than estimating the delay, a novel method is proposed to calculate the delay from the observer's phase response. In terms of system identification, the delay is compensated by aligning other measurements with the lagged estimate so that they have the same lag. The simulation results of an aero engine model show significant improvements in estimation. On one hand, the proposed approach improves the estimation accuracy, and on the other hand, it removes the assumption of zero delay and gives a new insight into the high-gain observer design.

Index Terms—High-gain observer, state estimation, parameter variation, time delay

I. INTRODUCTION

In many industrial processes, it is common that the system parameters vary due to aging and abrasion. For the purpose of condition monitoring and optimal control, it is important to have accurate knowledge of the parameter variations. For multiple-input multiple-output (MIMO) systems, some methods have been developed to estimate the parameter variations. Examples are the explicit parameter identification methods (e.g. *n4sid* [1], iterative gradient-based search method [2]) and observer-based model reference estimation algorithms (e.g. augmented observer [3], adaptive observers [4], [5], [6], and disturbance observer [7]).

As an augmented observer, a high gain observer (HGO) was recently proposed for fault estimation in [8] and then extended to estimate the parameter variations [9]. Under the assumption of bounded noises, the basic idea of the HGO-based estimation is the introduction of a new variable, referred to as "*disturbance*" that linearly depends on the concerned parameter variations and is augmented into the system state vector. Thus an augmented observer is designed to estimate both the system state and the disturbance simultaneously. Then the parameter variations are identified from the estimates of state and disturbance variables within an ARX (AutoregRessive eXogenous) framework [9]. In HGO, a high gain is chosen so that the bounded noises are significantly attenuated making the state/disturbance estimation errors as small as desired [8].

The identification performance of HGO depends on how good the disturbance estimate is. Although the estimation error, in terms of amplitude, can be made as small as desired by selecting a very

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high gain [8], [9], a time delay between the "actual" disturbance and its estimate may appear, due to the non-zero phase response of the observer. It is particularly true when a persistent excitation input is used for identification purpose. As proved in this paper, the parameter estimation error caused by the delay can make the estimation of parameter variations biased. The problem of delay was mitigated by using a very high gain (e.g. in the order of 10^9) [9], which reduces the delay to an ignorable level, but increases the computation complexity and encounters the problems of numerical instability.

Time Delay Estimation (TDE) [10] might be a solution. However, it is worth keeping in mind that, as a prediction-error identification method, the TDE may result in a combination of "wrong" delay and parameters, due to the fact that a wrong combination may give the best model approximation [11] [12] [13]. This often happens in non-integer time delay and/or low signal-to-noise ratio (SNR) scenarios. In the HGO, the time delay is nonlinear when the phase response is nonlinear and the state/disturbance estimation errors lead to a low SNR, which makes the TDE harder.

In contrast to estimating the delay, this paper proposes a method to calculate the delay of a given HGO. The delay is proved to be dependent on the observer gain, but be invariant to the parameter variations. This property enables a new way to mathematically and accurately calculate the delay in the frequency domain. As a result, the delay can be accurately aligned to reduce its impacts on the parameter identification. The HGO is formulated firstly in section (II) and the properties of estimation delay are analyzed in section (III), where it is proved that the delay is independent of parameter variations. In section (IV), a novel estimation scheme is proposed, where the delay is compensated (in the sense of identification) by lagging other variables by the same delay such that all the variables are aligned. As demonstrated in section (V), the application to a gas turbine engine model identification verifies the existence of estimation delay. The proposed delay alignment significantly improves the identification performance.

II. HIGH-GAIN OBSERVER DESIGN

A. High gain observer for state and disturbance estimation

Consider a continuous system described by

$$\begin{cases} \dot{x}(t) = (A_0 + \Delta A)x(t) + (B_0 + \Delta B)u(t) + \omega_i(t) \\ y(t) = Cx(t) + \omega_o(t) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the input, $y \in \mathbb{R}^p$ the output, $\omega_i(t) \in \mathbb{R}^n$ and $\omega_o(t) \in \mathbb{R}^p$ are the unknown-but-bounded input and output noises, respectively. A_0, B_0 are known matrices of the nominal model, ΔA and ΔB are the *parameter variations* to be identified. By introducing a new variable

$$d(t) = \Delta Ax(t) + \Delta Bu(t), \tag{2}$$

the system can be rewritten as

$$\begin{cases} \dot{x}(t) = A_0 x(t) + B_0 u(t) + d(t) + \omega_i(t) \\ y(t) = Cx(t) + \omega_o(t) \end{cases} . \tag{3}$$

In this study, $\dot{d}(t)$ is assumed to be bounded representing that ΔA and ΔB are slowly time-varying. The new variable $d(t)$ is referred to as "*disturbance*" to represent the parameter variations $[\Delta A, \Delta B]$. Interpreting $x(t)$ and $u(t)$ as the inputs and $d(t)$ as the output, the disturbance model (2) is a static 2-input-1-output ARX model. Since $u(t)$ is known, if $d(t)$ and $x(t)$ can be estimated, then $[\Delta A \ \Delta B]$ can be obtained. A high-gain observer (HGO) is recently proposed for estimating $x(t)$ and $d(t)$ simultaneously [8] [9].

Proposition 1: Denote $\bar{x}^T(t) = [x^T(t) \ d^T(t) \ \omega_o^T(t)] \in \mathbb{R}^{2n+p}$ and $\bar{A} = \begin{bmatrix} A_0 & I_n & 0 \\ 0 & 0_n & 0 \\ 0 & 0 & -I_p \end{bmatrix}$, $\bar{B} = \begin{bmatrix} B_0 \\ 0_{n \times m} \\ 0_{p \times m} \end{bmatrix}$, $\bar{C} = \begin{bmatrix} C & 0_{p \times n} & I_p \end{bmatrix}$, $\bar{E} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0_{p \times p} \end{bmatrix}$, $\bar{G} = \begin{bmatrix} I_n \\ 0_{n \times n} \\ 0_{p \times n} \end{bmatrix}$, $\bar{H} = \begin{bmatrix} 0_{n \times n} \\ I_{n \times n} \\ 0_{p \times n} \end{bmatrix}$, $\bar{N} = \begin{bmatrix} 0_{n \times p} \\ 0_{n \times p} \\ I_p \end{bmatrix}$, system (3) can be augmented into a descriptor system given by

$$\begin{cases} \bar{E}\dot{\bar{x}}(t) &= \bar{A}\bar{x} + \bar{B}u(t) + \bar{G}\omega_i(t) + \bar{H}\dot{d}(t) - \bar{N}\omega_o(t) \\ y(t) &= \bar{C}\bar{x}(t). \end{cases} \quad (4)$$

and its corresponding augmented observer can be formed as

$$\begin{cases} \bar{S}\dot{\xi}(t) = (\bar{A} - \bar{K}\bar{C})\xi(t) + \bar{B}u(t) - \bar{N}y(t) \\ \hat{\bar{x}}(t) = \xi(t) + \bar{S}^{-1}\bar{L}y(t) \end{cases} \quad (5)$$

where $\bar{S} = \bar{E} + \bar{L}\bar{C}$. Note that $\overline{K}$ and $\overline{L} \in \mathbb{R}^{(2n+p) \times n}$ are the gain matrices to be designed.

For more details of the derivation, it is suggested to refer to [9] and [8]. As proved in [8], if (A_0, C) is observable, the augmented system is observable if and only if

$$rank \begin{bmatrix} A_0 & I_n \\ C & 0 \end{bmatrix} = 2n. \quad (6)$$

When condition (6) is met, a stable high-gain observer (5) exists with the gain matrix

$$\bar{K} = \bar{S}\bar{P}^{-1}\bar{C}^T, \quad (7)$$

where $\bar{P}$ is solved from the Lyapunov equation $-(\mu I + \bar{S}^{-1}\bar{A})^T\bar{P} - \bar{P}(\mu I + \bar{S}^{-1}\bar{A}) = -\bar{C}^T\bar{C}$ with $\mu > 0$ satisfying the real part $\Re[\lambda_i(\bar{S}^{-1}\bar{A})] > -\mu, \forall i \in \{1, 2, ..., 2n + p\}$. The state x and disturbance d can now be estimated simultaneously as $\hat{x}(t) = [I_n \ 0_n \ 0_{n \times p}]\hat{\bar{x}}(t)$ and $\hat{d}(t) = [0_n \ I_n \ 0_{n \times p}]\hat{\bar{x}}(t)$. For the sake of notation, time index t may be dropped in the following textual description.

B. Estimation of Parameter variations

Define estimation errors $e(t) = \bar{x}(t) - \hat{\bar{x}}(t)$, $e_x(t) = x(t) - \hat{x}(t)$ and $e_d(t) = d(t) - \hat{d}(t)$, substituting $x = \hat{x} + e_x$ and $d = \hat{d} + e_d$ into (2) gives

$$\hat{d}(t) = \Delta A\hat{x}(t) + \Delta Bu(t) + \Delta Ae_x(t) - e_d(t) \quad (8)$$

Here, $(\hat{d}, \hat{x}, u)$ is regarded as the observation of the ARX model and $\Delta Ae_x(t) - e_d(t)$ is the observation noise. The impacts of w_i and w_o are represented by $e_x(t)$ and $e_d(t)$.

Theorem 1: If the original plant (1) is observable and condition (6) is met, given bounded noises ω_i and ω_o, there exists an observer (5) and associated gain matrix $\bar{K}$ (7) that can make the steady estimate errors $e_x(t)$ and $e_d(t)$ as small as desired.

Proof: By comparing our system (4) with the system model in [8] and interpreting $d(t)$ (2) as the fault signal, it is easy to verify that the value of α, B_a and D_a in [8] are $\alpha = 0$, $B_a = I_n$ and $D_a = 0$. Thus, if condition (6) is met, according to Theorem 1 in [8], then a gain matrix $\bar{K}$ exists and the estimation error is bounded and the bound can be made as small as desired. ∎

Theorem 2: If both $e_d(t)$ and $e_x(t)$ tend to desired small values as time approaches infinity, minimizing the mean square error of model

(8) makes $[\Delta\hat{A}, \Delta\hat{B}]$ approach $[\Delta A, \Delta B]$, but subject to a bias of up to $\gamma\beta\frac{\sum_{k=1}^{M}|e_i(t)|}{M}$.

Proof: From the purpose of analysis, the disturbance model (8) in matrix form can be rewritten as a set of multi-input-single-output submodels in vector form:

$$\hat{d}_i(t) = \varphi^T(t)\theta_i^* + e_i(t) \quad i = \{1, 2, \ldots, n\} \quad (9)$$

where $\varphi(t) = \begin{bmatrix} \hat{x}(t) \\ u(t) \end{bmatrix}$ is the regression vector, $\theta_i^* = [\Delta A_i \ \Delta B_i]^T \in \mathbb{R}^{1 \times (n+m)}$ is the true value to be found, $[\Delta A_i \ \Delta B_i]$ represents the i-th row of the matrix $[\Delta A \ \Delta B]$ and $e_i(t) = \Delta A_i e_x(t) - e_{d,i}(t)$, where $e_{d,i}(t)$ is the i-th element of $e_d(t)$. The submodel (9) is a zero order $(n + m)$-input-one-output ARX model and can be identified by using the Least Square Estimation (LSE) in the discrete domain. Note that the order of output $d_i(t)$ is zero.

As the observer works in the continuous time domain, a sampling process has to be applied to the continuous data $[\hat{d}(t), \hat{x}(t), u(t)]$ before identification. Assume M sampled data pairs $[\hat{d}(kT), \hat{u}(kT), u(kT)]$, $k = 1, 2, ..., M$ have been collected at a sampling interval of T second, the LSE method is to find an optimal $\hat{\theta}_i$ by minimizing the mean squared error function $V^{(M)}(\theta_i) = \frac{1}{M}\sum_{k=1}^{M}(\hat{d}_i(KT) - \varphi^T(kT)\theta_i)^2$. The parameter estimated by the LSE is

$$\hat{\theta}_i^{(M)} = R(M)^{-1}\Big[\sum_{k=1}^{M}\varphi(kT)[\varphi(kT)^T\theta_i^* + e_i(kT)]\Big]$$
$$= \theta_i^* + R(M)^{-1}\Big[\sum_{k=1}^{M}\varphi(kT)e_i(kT)\Big] \quad (10)$$

where $R(M) = \sum_{k=1}^{M}\varphi(kT)\varphi^T(kT)$. Therefore, the estimation error $\tilde{\theta}_i$ is

$$\tilde{\theta}_i = \hat{\theta}_i^{(M)} - \theta_i^* = R(M)^{-1}\Big[\sum_{k=1}^{M}\varphi(kT)e_i(kT)\Big] \quad (11)$$

and the mathematical expectation of $\tilde{\theta}_i$ is

$$E(\tilde{\theta}_i) = E\Big[R(M)^{-1}\sum_{k=1}^{M}\varphi(kT)e_i(kT)\Big] \quad (12)$$

Note that e_i is the estimation error depending on the variable u and hence φ. Further, both the estimate $\hat{x}$ and the error e_i are correlated with the input noises ω_i hence φ correlates with e_i. Therefore $E\big[\sum \varphi(kT)e_i(kT)\big]$ is not zero. However, u is a bounded deterministic signal and the input/output noises are bounded as well. Hence φ and $R(M)$ are bounded matrices. Following the terminology introduced in [14], assuming $|\varphi| < \beta$ and $||R(M)^{-1}|| < \gamma$ and according to the definition of matrix norm, the absolute value of the expectation of $\tilde{\theta}_i$ is governed by

$$E(|\tilde{\theta}_i|) = E(||R(M)^{-1}|| \cdot |\sum_{k=1}^{M}\varphi(kT)e_i(kT)|)$$
$$< E(\gamma\sum_{k=1}^{M}\beta|e_i(kT)|) \quad (13)$$

where $|| \cdot ||$ denotes matrix's Euclidean norm and $| \cdot |$ denotes vector Euclidean norm. Thus

$$E(|\tilde{\theta}_i|) < \gamma\beta\frac{\sum_{k=1}^{M}|e_i(kT)|}{M}. \quad (14)$$

According to Theorem 1, e_x and e_d can be made as small as desired by selecting a gain matrix $\bar{K}$ appropriately. Hence $e_i(t)$ is as small as desired. Therefore, $E(|\tilde{\theta}_i|)$ can be made as small as desired, but not necessarily zero. It implies that minimizing the mean square error $V^{(M)}(\theta_i)$ of the ARX model (9) makes $[\Delta\hat{A}, \Delta\hat{B}]$ approach

to $[\Delta A, \Delta B]$, but a bias of up to $\gamma\beta\frac{\sum_{k=1}^{M}|e_i(kT)|}{M}$ may exist. This completes the proof. ∎

C. Time-delay of HGO

Theorem 1 shows that, in the steady state, $[\hat{d}, \hat{x}]$ approaches to $[d, x]$ asymptotically. Thus, $e_i(t)$ approximates to zero and the bias of the estimation of $[\Delta A_i \ \Delta B_i]$ is close to zero. However, due to the dynamic response of the HGO, an estimation delay may exist between a variable and its estimate and it depends on the response speed of the observer. The delay may cause a relatively large $e_i(t)$ with the consequence of a significant bias. In particularly, in order to cover the whole frequency range of interest. the parameter identification usually requires persistent excitation rather than a step signal. Thus an obvious delay will appear.

More specifically, the HGO can be regarded as a filter with input $d(t)$ and output $\hat{d}(t)$. And the dynamics of the HGO can be described by a transfer function matrix (TFM) $F(s)$ with phase response $\Phi(j\omega)$. Then, the value

$$\tau(\omega) = -\frac{\Phi(j\omega)}{\omega} \tag{15}$$

gives the *"time delay"* between the input and output signals. $\tau(\omega)$ is also referred to as phase delay, pure delay or transport lag. Generally, $\tau(\omega)$ is a function of frequency ω and may vary with respect to frequency. According to our preliminary experiments, the delay between $x(t)$ and $\hat{x}(t)$ is tiny and negligible. Thus only the disturbance's delay is considered in this note.

III. TIME DELAY IN DISTURBANCE ESTIMATION

In order to derive the mathematical expression of $\tau(\omega)$ regarding $d(t)$ to $\hat{d}(t)$, some TFMs of the HGO are first formulated, then the relationship between $d(t)$ and $\hat{d}(t)$ is presented.

From the augmented system (5), the TFM $G_{\hat{d}u}(s)$ relating $u(t)$ to $\hat{d}(t)$ can be obtained:

$$G_{\hat{d}u}(s) = G_1(s) + G_2(s)G_{yu}(s) + G_3(s)G_{\dot{y}u}(s). \tag{16}$$

where

$$G_1(s) = [0_n \ I_n \ 0_{n\times p}] \cdot [s\bar{S} - (\bar{A} - \bar{K}\bar{C})]^{-1}\bar{B} \tag{17}$$

$$G_2(s) = [0_n \ I_n \ 0_{n\times p}] \cdot (s\bar{S} - (\bar{A} - \bar{K}\bar{C}))^{-1}\bar{K} \tag{18}$$

$$G_3(s) = [0_n \ I_n \ 0_{n\times p}] \cdot (s\bar{S} - (\bar{A} - \bar{K}\bar{C}))^{-1}\bar{L} \tag{19}$$

$$G_{yu}(s) = C[sI - (A_0 + \Delta A)]^{-1}(B_0 + \Delta B) \tag{20}$$

$$G_{\dot{y}u}(s) = C(A_0 + \Delta A)[sI - (A_0 + \Delta A)]^{-1}(B_0 + \Delta B) \\ + C(B_0 + \Delta B) \tag{21}$$

Recalling the plant (1) with disturbance $d(t) = \Delta Ax(t) + \Delta Bu(t)$, it is easy to verify that the TFM $H_{du}(s)$ relating $u(t)$ to $d(t)$ is

$$H_{du}(s) = \Delta A[sI - (A_0 + \Delta A)]^{-1}(B_0 + \Delta B) + \Delta B \tag{22}$$

For the sake of notation, some abbreviations are first defined as $\Lambda = (sI - A_0)$, $\Psi = [sI - (A_0 + \Delta A)]$ and

$$F(s) = G_2(s)C\Lambda^{-1} + G_3(s)[CA_0\Lambda^{-1} + C]. \tag{23}$$

It is easy to verify $F(s)$ is a $n \times n$ TFM. In this subsection, a constructive proof of $G_{\hat{d}d}(s) = F(s)$ is presented, where $G_{\hat{d}d}(s)$ is the TFM relating $d(t)$ to its estimate $\hat{d}(t)$. Before giving the main theorem, two lemmas are proved as follows.

Lemma 1: Given the plant (A_0, B_0, C), if (A_0, C) is observable and condition (6) is met, then $H_1(s) = 0$. Here $H_1(s)$ is a TFM

$$H_1(s) = G_1(s) + G_2(s)[C\Lambda^{-1}B_0] + G_3(s)[CA_0\Lambda^{-1}B_0 + CB_0]$$

Proof: This can be proved by analyzing the dynamics of the high-gain observer. The plant system (with parameter variations) and its associated observer can be decomposed as shown in Figure 1, where the plant output $y(t)$ is split into two parts: $y_m(t)$ from the
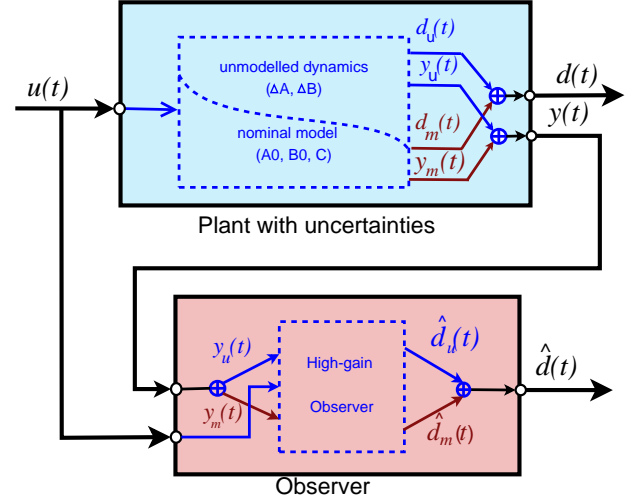

Fig. 1. Decomposition of the plant and its high-gain observer.

nominal model (A_0, B_0, C) and $y_u(t)$ from the unmodeled dynamics. The same applies to $d(t)$.

$$y(t) = y_m(t) + y_u(t), \qquad d(t) = d_m(t) + d_u(t) \tag{24}$$

where $y(t)$ is measurable, and $d(t)$ is unmeasurable. Note that, since no model uncertainty exists within the nominal model (A_0, B_0, C), $d_m(t)$ is always zero $d_m(t) \equiv 0$. According to the additionality of linear systems, the outputs of the high-gain observer can be decomposed in a similar way:

$$\hat{y}(t) = \hat{y}_m(t) + \hat{y}_u(t), \qquad \hat{d}(t) = \hat{d}_m(t) + \hat{d}_u(t) \tag{25}$$

where $\hat{d}_m(t)$ is the estimate of $d_m(t)$ and $\hat{d}_u(t)$ is the estimate of $d_u(t)$.

Observe that the terms in $H_1(s)$ have a special meaning: $C\Lambda^{-1}B_0$ is the TFM relating $u(t)$ to $y_m(t)$ and $CA_0\Lambda^{-1}B_0 + CB_0$ is the TFM relating $u(t)$ to $\dot{y}_m(t)$. According to (16), it can be verified that $H_1(s)$ is exactly the TFM relating $u(t)$ to $\hat{d}_m(t)$:

$$\hat{d}_m(s) = H_1(s)u(s) \tag{26}$$

where $\hat{d}_m(s)$ and $u(s)$ are s-transforms of $\hat{d}_m(t)$ and $u(t)$, respectively.

Furthermore, if (A_0, C) is observable and the condition (6) is satisfied, then the designed observer is stable and the disturbance estimate $\hat{d}(t)$ is asymptotically stable. With the decomposition as shown in Figure 1, it can be stated that $\hat{d}_m(t)$ approaches $d_m(t)$ asymptotically. Since $d_m(t) \equiv 0$, whatever the values of $u(t)$ and $[\Delta A \ \Delta B]$ are, $\hat{d}_m(t)$ is always zero. That is

$$H_1(s)u(s) \equiv 0 \text{ for any } u(s). \tag{27}$$

It implies that $\hat{d}_m(t)$ has nothing to do with $u(t)$ and (27) is satisfied if and only if $H_1(s)$ is a zero matrix. ∎

Lemma 2: Given the plant (A_0, B_0, C), if (A_0, C) is observable and condition (6) is met, then $G_1(s) + G_2(s)[C\Psi^{-1}B_0] + G_3(s)[CA_0\Psi^{-1}B_0 + CB_0] + G_3(s)[C\Delta A\Psi^{-1}B_0] = F(s) \cdot \Delta A^{-1}\Psi^{-1}B_0$ and $G_2(s)C\Psi^{-1}\Delta B + G_3(s)C[A_0\Psi^{-1} + I]\Delta B + G_3(s)C\Delta A\Psi^{-1}\Delta B = F(s) \cdot (\Delta A\Psi^{-1}\Delta B + \Delta B)$.

The proofs of Lemma 2 can be done simply by algebraic manipulations.

Theorem 3: Given a system (A_0, B_0, C) with parameter variation $[\Delta A, \ \Delta B]$ and the corresponding high-gain observer (5), for any value of $[\Delta A, \ \Delta B]$, the TFM $G_{\hat{d}d}(s)$ relating $d(t)$ to $\hat{d}(t)$ is equivalent to $F(s)$

$$G_{\hat{d}d}(s) = F(s), \tag{28}$$

and the time delay τ between $d(t)$ and its estimate $\hat{d}(t)$ is independent of $[\Delta A, \Delta B]$.

Proof: Substituting $\Psi = sI - (A_0 + \Delta A)$, $\Lambda = sI - A_0$ into $G_{\hat{d}u}(s)$ (16) gives

$$\begin{aligned} G_{\hat{d}u}(s) = &\underline{G_1(s)} + G_2(s)C\Psi^{-1}B_0 + G_2(s)C\Psi^{-1}\Delta B \\ &+ \underline{G_3(s) \cdot C[A_0\Psi^{-1} + I]B_0} + \underline{G_3(s) \cdot C\Delta A\Psi^{-1}B_0} \\ &+ G_3(s) \cdot C[A_0\Psi^{-1} + I]\Delta B + G_3(s) \cdot C\Delta A\Psi^{-1}\Delta B \end{aligned} \tag{29}$$

According to Lemma 2, it is easy to verify that

$$G_{\hat{d}u}(s) = F(s)\cdot(\Delta A^{-1}\Psi^{-1}B_0) + F(s)\cdot(\Delta A\Psi^{-1}\Delta B + \Delta B) \tag{30}$$

According to the definition of $H_{du}(s)$ as given by (22) and the definition of Ψ, the equation above becomes $G_{\hat{d}u}(s) = F(s)\cdot H_{du}(s)$. Then, one has

$$\hat{d}(s) \quad = \quad G_{\hat{d}u}(s)u(s) = F(s)H_{du}(s)u(s) = F(s)d(s) \tag{31}$$

Equation (31) indicates that $F(s)$ is the TFM relating $d(t)$ to $\hat{d}(t)$, that is $G_{\hat{d}d}(s) = F(s)$.

From the definition of $F(s)$ in (23), it is easy to verify that $G_{\hat{d}d}(s)$ is determined by $G_2(s)$, $G_3(s)$ and (A_0, B_0, C) only. Since both $G_2(s)$ and $G_3(s)$ are independent of $[\Delta A, \Delta B]$, $G_{\hat{d}d}(s)$ does not depend on $[\Delta A, \Delta B]$ as well. Consequently, the time delay τ between $d(t)$ and its estimate $\hat{d}(t)$ is also independent of $[\Delta A, \Delta B]$. This completes the proof of Theorem 3. ■

The TFM $F(s)$ is a $n \times n$ matrix and can be expressed in terms of magnitude response matrix $M(s)$ and phase response matrix $\Phi(s)$. Ideally, $M(s)$ is an identity matrix and $\Phi(s)$ is a zero matrix. In practice, $M(s)$ approximates to an identity matrix, thus $\hat{d}_i(t)$ is dominantly determined by $d_i(t)$. In the calculation of the delay τ, only the diagonal elements of $\Phi(s)$ are considered. Substituting $s = \mathbf{j}\,\omega$ into $\Phi(s)$, the delay function $\tau_i(\omega)$ is

$$\tau_i(\omega) = -\frac{\Phi_{ii}(\mathbf{j}\omega)}{\omega} \tag{32}$$

where $\tau_i(\omega)$ is the delay between $\hat{d}_i(t)$ and $d_i(t)$ at frequency ω, and Φ_{ii} the i-th diagonal element of Φ.

Remark 1. Theorem 3 implies that the time delay $\tau_i(\omega)$ is invariant under different values of $[\Delta A, \Delta B]$. Hence, it is practical to compute the delay by assigning an arbitrary value to $[\Delta A, \Delta B]$, and the calculated delay is applicable to any values of $[\Delta A, \Delta B]$.

Remark 2. The time delay $\tau_i(\omega)$ computed from the phase response matrix $\Phi(s)$ is more accurate than that of TDE methods. Furthermore, Theorem 3 gives the TFM $F(s)$ which make it possible to design a phase-shifter to lag other variables by the same delay.

Remark 3. The phase response can be used as a criteria to evaluate the performance of the high-gain observer. Generally, $\Phi(j\omega)$ and $\tau(\omega)$ vary as the frequency changes. From the perspective of observer design, an desired observer is one with zero delay or a linear $\Phi(j\omega)$ over the frequency range of interest.

IV. ESTIMATION OF PARAMETER VARIATIONS WITH DELAY ALIGNMENT

In the context of parameter identification, the purpose of delay alignment is to align the data pairs $[\hat{x}^T(t) \ u^T(t) \ \hat{d}_i^T(t)]$ with respect to $[x^T(t) \ u^T(t) \ d_i^T(t)]$ in the ARX model (9). Since $\hat{d}_i(t)$ has been lagged behind $d_i(t)$ by $\tau_i(\omega)$, either leading $\hat{d}_i(t)$ or lagging $[\hat{x}^T(t), u^T(t)]$ by $\tau_i(\omega)$ can align the data pairs. Due to the causality of the practical system, leading $\hat{d}(t)$ is impossible. The fundamental of the proposed delay alignment approach is to make use of a set of "phase-shift" filters $\{L_i(s)\}$ whose phase responses are equivalent to $\{\Phi_{ii}(s)\}$ respectively. By this way, $\hat{x}^T(t)$ and $u^T(t)$ are delayed by the same time τ_i. Thus, the data pairs are aligned as $[\hat{x}^T(t - \tau_i) \ u^T(t - \tau_i)], \hat{d}_i^T(t)$.
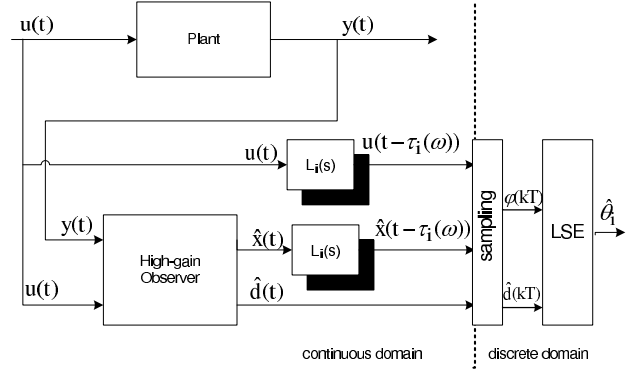

Fig. 2.   The scheme of delay alignment and parameter estimation

The key step in delay alignment is to choose the coefficients of the compensation filter $L_i(s)$ properly to obtain the desired phase-delay frequency response. Since the TFM relating $\hat{d}(t)$ to $d(t)$ has been given in Theorem 3, one straightforward choice is to use the diagonal elements of $F(s)$ as $\{L_i(s)\}$. That is $L_i(s) = F_{ii}(s)$.

Figure 2 illustrates this scheme. As the plant and HGO work in the continuous time domain and the LSE is a discrete identification method, a sampling process is applied on the aligned signals $\hat{x}(t-\tau_i)$, $u(t-\tau_i)$ and $\hat{d}(t)$. The LSE uses the sampled data pairs to estimate $[\Delta A, \Delta B]$. As shown in Theorem 2, the ARX model to be identified is converted into a set of zero order $(n+p)$-input-single-output ARX models.

V. SIMULATION AND RESULTS

In this section, the proposed scheme is applied to a simulated gas turbine engine with real input signal collected at the engine test bed. In order to illustrate the time delay clearly, a noise free case is first presented. A reduced third order linear model of a turbojet engine is taken from [15] and represented with the nominal coefficient matrices $A_0 = \begin{bmatrix} -112.27 & 52.92 & 42.24 \\ -48.120 & 0.0 & 47.41 \\ 2.838 & -1.258 & -4.09 \end{bmatrix}$, $B_0 = [32.42 \ \ 0.6685 \ \ 0.0]^T$ and C is an identity matrix. These coefficient matrices $[A_0 \ B_0]$ are accurate when the engine is in healthy condition. Due to various reasons (e.g. aging), ΔA and ΔB become non-zero matrices. In the simulation, the parameter matrices change at 10 second as follows:

$$\Delta A = \begin{bmatrix} -11.227 & 5.292 & 4.224 \\ -5.900 & 3.145 & 4.741 \\ -5.830 & -3.90 & 5.23 \end{bmatrix}, \Delta B = \begin{bmatrix} 6.484 \\ 1.534 \\ 4.00 \end{bmatrix}, t > 10s$$

A. Time Delay Analysis

In order to illustrate the estimation delay, a multi-sine wave over frequency $\Omega = [0, 5]$ Hz is fed to the engine model as the input. The time delays $\tau_i(\omega)$ (32) calculated by the proposed algorithm

are illustrated in the left column of Fig. 3. It shows that the delays decrease in a non-linear fashion as the frequency increases. Over the frequency $[0, 2]$ (Hz), $\tau_1(\omega)$ is $6.48ms$, and both $\tau_2(\omega)$ and $\tau_3(\omega)$ are around $6.0ms$.

The simulation results of the disturbance estimation delay are shown in the right column of Fig. 3. The values of the delays ($\tau_1 = 6.5\ ms$, $\tau_2 = 6.0\ ms$ and $\tau_3 = 6.0ms$, read from the simulation) match the delays calculated from $\Phi_{ii}(\omega), (i = 1, 2, 3)$.

B. Noise-corrupted case with real input data

In this section, the real data with main energy over the frequency range of $[1, 2]$ Hz is used as the input. The system is assumed subjected to both input noise $\omega_i = 0.4sin(20t) + n_i(t)$ and small output noise $\omega_o = 0.02sin(5t) + n_o(t)$, where $n_i(t)$ is a white noise with zero mean and the variance of 0.001; $n_o(t)$ is a white noise with zero mean and the variance of 0.0001.

The identified parameter variations are shown in Fig. 4 (with delay alignment) and Fig. 5 (without delay alignment). The proposed delay alignment shows good estimation results and, contrarily, the estimation approach without delay alignment fails. In Fig. 5, the variations and errors are so large that the method without delay alignment does not make any sense.

Compared with the results in [9], where no delay alignment technique was used and a very high gain had to be adopted (at the order of 10^{10}) to achieve an acceptable result, the proposed delay alignment method is able to achieve a similar performance at a relatively small gain (at the order of 10^6). The advantage is that a smaller gain is numerically more stable and provides better computation efficiency in practice.

(a) Estimates of $[\Delta A\ \Delta B]$ (with delay alignment)

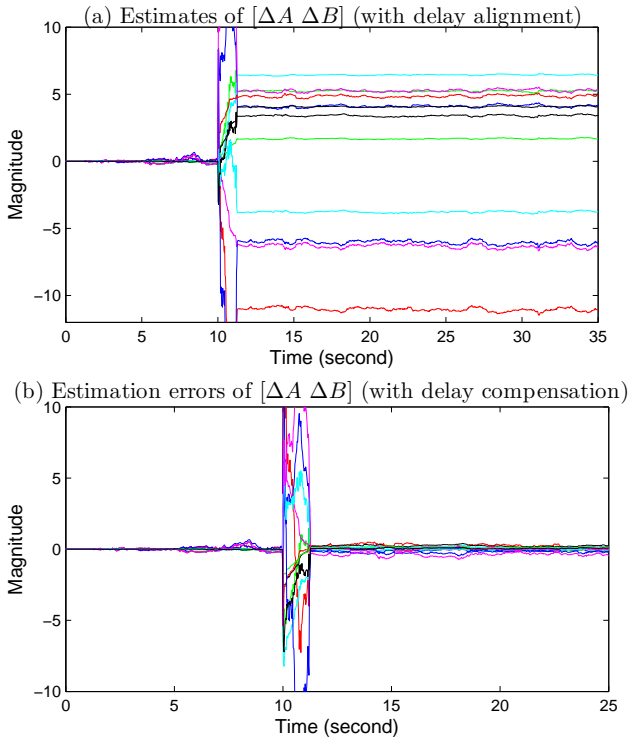(b) Estimation errors of $[\Delta A\ \Delta B]$ (with delay compensation)

Fig. 4. The parameter variation $[\Delta \hat{A}, \Delta \hat{B}]$ estimated with delay alignment (using real input and corrupted by noises)

The estimated $[\Delta \hat{A} | \Delta \hat{B}]$ are listed in the second column of Table I. For comparison, *n4sid* is also evaluated, where only the input-output pairs $[u, y]$ are used to identify the parameters. Then the parameter variations are calculated as the parameter estimates
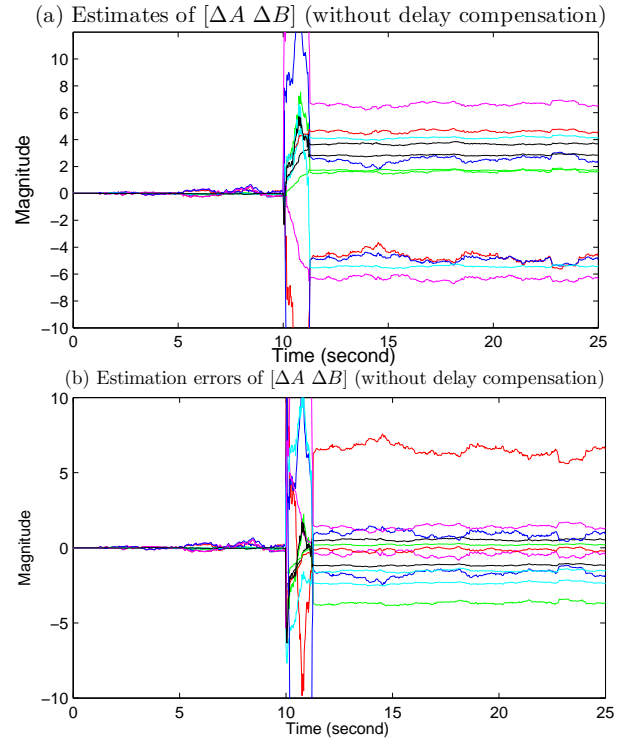

(a) Estimates of $[\Delta A\ \Delta B]$ (without delay compensation)

(b) Estimation errors of $[\Delta A\ \Delta B]$ (without delay compensation)

Fig. 5. The parameter variation $[\Delta \hat{A}, \Delta \hat{B}]$ estimated without delay alignment (using real input and corrupted by noises)

minus their known nominal values. The third column shows the estimation errors in percentage. The percentage error is defined as $|(a_{ij} - \hat{a}_{ij})/a_{ij}| \times 100\%$ and $|(b_j - \hat{b}_j)/b_j| \times 100\%$, where a_{ij}, b_j are the elements of ΔA, ΔB and $\hat{a}_{ij}$ and $\hat{b}_j$ are their estimates, respectively. It can be seen that the proposed delay alignment scheme improves the parameter identification performance.

VI. CONCLUSION

Due to the dynamics of HGO and its nonzero phase response, the time delay appears which affects the performance of parameter identification. This paper analyzes the properties of disturbance estimation in the HGO and proves that the estimation delay is independent of the parameter variations. Thus the delay can be computed accurately from the TFMs and be compensated (in the sense of identification) by aligning all the variables by the same delay. When the delay is presented in the HGO, the proposed algorithm improves the performance significantly. This has been verified by the simulation results on a gas turbine engine model.

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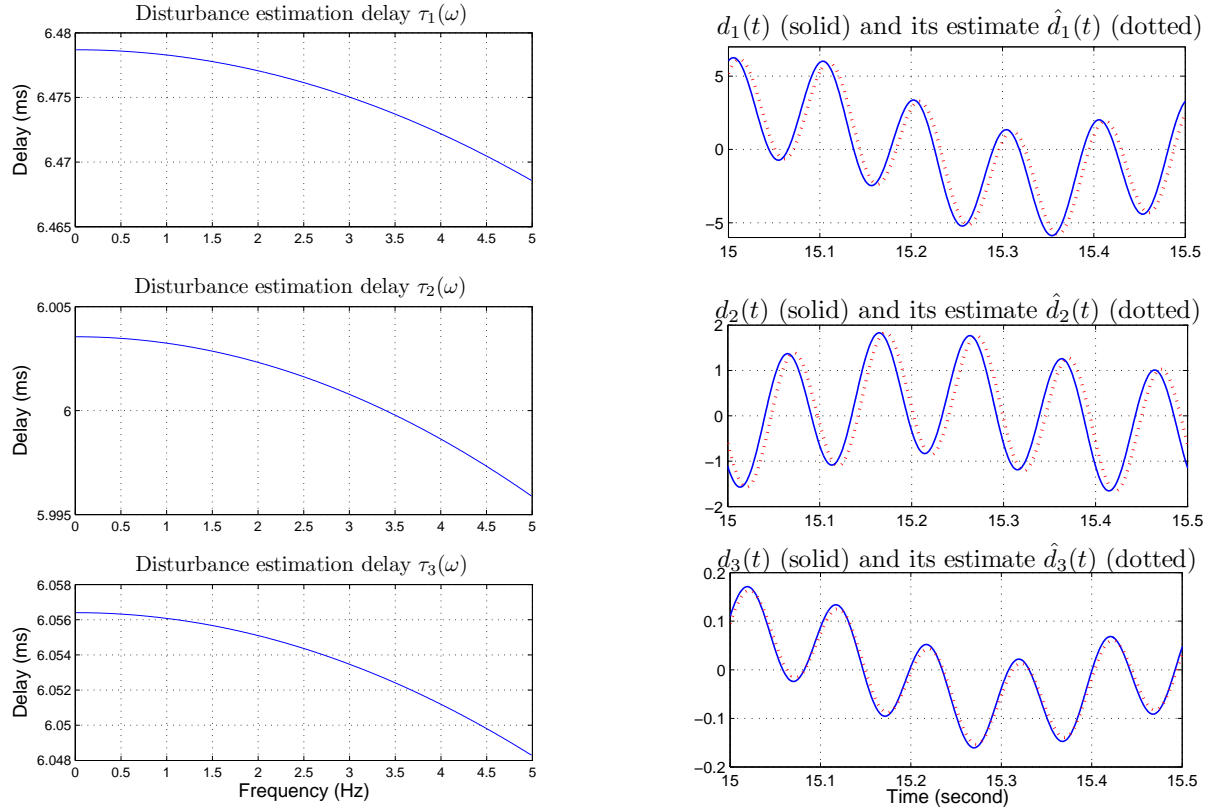
Fig. 3.　Left, time delay $\tau_i(\omega)$ of $\hat{d}_i(t)$; right, Disturbances $d_i(t)$ and their corresponding estimates $\hat{d}_i(t)$ ($i = 1, 2, 3$).

TABLE I
COMPARISON OF THE PARAMETER ESTIMATES (WITH REAL INPUT AND NOISES)

	estimates $[\Delta\hat{A} \mid \Delta\hat{B}]$				Estimation error (%)			
with delay alignment†	-10.9922	5.2341	4.1550	6.3645	2.1%	1.1%	1.6%	1.8%
	-6.3315	3.3852	4.8709	1.6769	7.3%	7.6%	2.7%	9.3%
	-5.9757	-3.7537	5.3266	3.9677	2.5%	3.8%	1.9%	0.8%
without delay alignment †	-4.9671	1.7047	2.7120	4.1446	55.8%	67.8%	35.8%	36.1%
	-6.4244	3.6901	4.6885	1.7594	8.9%	17.3%	1.1%	14.7%
	-5.1441	-5.3051	6.8089	2.8295	11.8%	36.0%	30.2%	29.3%
n4sid method†	-10.9833	5.1733	4.1309	-6.7586	7.3%	13.3%	20.6%	3.19%
	-5.7203	3.0576	4.6712	-2.7998	19.1%	32.9%	25.8%	6.87%
	-5.5367	-3.9953	5.1562	3.8933	20.6%	28.9%	25.2%	9.04%

†Note: *n4sid* uses a sampling rate of 1kHz, other two methods use 40 Hz.

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